Chapter 7.
Inferences about Population Variances
Introduction (1)

- The variability of a population’s values is as important as the population mean.
- Hypothetical distribution of *E. coli* concentrations from two analytical methods: petrifilm HEC test and hydrophobic grid membrane filtration (HGMF).
**Introduction (2)**

- To compare both the means and standard deviation.
  - To use HEC and HGMF procedures to 24 pure culture samples each.
  - To apply both procedures to artificially contaminated beef samples.

### Table 7.1 E. coli readings (log10 (CFU/ml) from HGMF and HEC methods.

<table>
<thead>
<tr>
<th>Sample</th>
<th>HGMF</th>
<th>HEC</th>
<th>Sample</th>
<th>HGMF</th>
<th>HEC</th>
<th>Sample</th>
<th>HGMF</th>
<th>HEC</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6.65</td>
<td>6.67</td>
<td>9</td>
<td>6.89</td>
<td>7.08</td>
<td>17</td>
<td>7.07</td>
<td>7.25</td>
</tr>
<tr>
<td>2</td>
<td>6.62</td>
<td>6.75</td>
<td>10</td>
<td>6.90</td>
<td>7.09</td>
<td>18</td>
<td>7.09</td>
<td>7.28</td>
</tr>
<tr>
<td>3</td>
<td>6.68</td>
<td>6.83</td>
<td>11</td>
<td>6.92</td>
<td>7.09</td>
<td>19</td>
<td>7.11</td>
<td>7.34</td>
</tr>
<tr>
<td>4</td>
<td>6.71</td>
<td>6.87</td>
<td>12</td>
<td>6.93</td>
<td>7.11</td>
<td>20</td>
<td>7.12</td>
<td>7.37</td>
</tr>
<tr>
<td>5</td>
<td>6.77</td>
<td>6.95</td>
<td>13</td>
<td>6.94</td>
<td>7.11</td>
<td>21</td>
<td>7.16</td>
<td>7.39</td>
</tr>
<tr>
<td>6</td>
<td>6.79</td>
<td>6.98</td>
<td>14</td>
<td>7.03</td>
<td>7.14</td>
<td>22</td>
<td>7.28</td>
<td>7.45</td>
</tr>
<tr>
<td>7</td>
<td>6.79</td>
<td>7.03</td>
<td>15</td>
<td>7.05</td>
<td>7.14</td>
<td>23</td>
<td>7.29</td>
<td>7.58</td>
</tr>
<tr>
<td>8</td>
<td>6.81</td>
<td>7.05</td>
<td>16</td>
<td>7.06</td>
<td>7.23</td>
<td>24</td>
<td>7.30</td>
<td>7.54</td>
</tr>
</tbody>
</table>
Introduction (3)

- The two procedures appear to be very similar with respect to the width of box and length of whiskers, but HEC has a larger median than HGMF.
- Also, the variability in concentration readings for HEC appears to be slightly greater than that of HGMF.
The initial conclusion might be that the two procedures yield different distributions of readings for their determination of *E. coli* concentrations.

However, we need to determine whether the differences in their sample means and standard deviations infer a difference in the corresponding population values.

Inferential problems about population variances are similar to the problems addressed in making inferences about the population mean. We must construct point estimators, confidence intervals, and the test statistics from the randomly sampled data to make inferences about the variability in the population values.
Estimation and Tests for a Population Variance (1)

• Unbiased estimator

\[ s^2 = \frac{\sum (y - \bar{y})^2}{n - 1} \]

- \( s^2 \) is an unbiased estimator of \( \sigma^2 \).
- If the population distribution is normal, then the sampling distribution of \( s^2 \) can have a shape similar to those depicted in Figure 7.3.
- It can be shown that the statistic \( \frac{(n-1)s^2}{\sigma^2} \) follows a chi-square distribution with \( df = n-1 \). The mathematical formula for the chi-square (\( \chi^2 \)) probability distribution is very complex so it is not displayed here.
Estimation and Tests for a Population Variance (2)

**FIGURE 7.3**
Densities of the chi-square (df = 5, 15, 30) distribution
Properties of Chi-Square Distribution (1)

- The chi-square distribution is positively skewed with values between 0 and $\infty$.
- There are many chi-square distributions and they are labeled by the parameter degrees of freedom (df).
- The mean and variance of the chi-square distribution has $\text{df} = 30$, then the mean and variance of that distribution are $\mu = 30$ and $\sigma^2 = 60$.
- Because the chi-square distribution is not symmetric, the confidence intervals based on this distribution do not have the usual form, estimate $\bar{x}$ error, as we saw for $\mu$ and $\mu_1 - \mu_2$. 
Properties of Chi-Square Distribution (2)

- The $100(1 - \alpha)\%$ confidence interval for $\sigma^2$ is obtained by dividing the estimator of $\sigma^2$, $s^2$, by the lower and upper $\alpha/2$ percentile, $\chi^2_L$ and $\chi^2_U$.

$$\frac{(n-1)s^2}{\chi^2_L} < \sigma^2 < \frac{(n-1)s^2}{\chi^2_U}$$
Statistical Test for $\sigma^2$

$H_0$: 1. $\sigma^2 \leq \sigma_0^2$
2. $\sigma^2 \geq \sigma_0^2$
3. $\sigma^2 = \sigma_0^2$

$H_a$: 1. $\sigma^2 > \sigma_0^2$
2. $\sigma^2 < \sigma_0^2$
3. $\sigma^2 \neq \sigma_0^2$

T.S.: $\chi^2 = \frac{(n-1)s^2}{\sigma_0^2}$

R.R.: For a specified value of $\alpha$,
1. Reject $H_0$ if $\chi^2$ is greater than $\chi^2_U$, the upper-tail value for $a = \alpha$ and df = $n - 1$.
2. Reject $H_0$ if $\chi^2$ is less than $\chi^2_L$, the lower-tail value for $a = 1 - \alpha$ and df = $n - 1$.
3. Reject $H_0$ if $\chi^2$ is greater than $\chi^2_U$, based on $a = \alpha/2$ and df = $n - 1$, or less than $\chi^2_L$, based on $a = 1 - \alpha/2$ and df = $n - 1$.

Check assumptions and draw conclusions.
Note 1

• When sample sizes are moderate to large (n ≥ 30), the t distribution-based procedures can be used to make inferences about $\mu$ even when the normality condition does not hold, because for moderate to large sample sizes the Central Limit Theorem provide that the sampling distribution of the sample mean is approximately normal.

• Unfortunately, the same type of result does not hold for the chi-square-based procedures for making inference about $\sigma^2$; that is, if the population distribution is distinctly nonnormal, then these procedures for $\sigma^2$ are not appropriate even if the sample size is large.

• If a boxplot or normal probability plot of the sample data shows substantial skewness or a substantial number of outliers, the chi-square-based inference procedures should not be applied.
Example 7.1

- To test whether the potency of a specific pesticide meets the potency claimed by the manufacturer --- the drop in potency from 0 to 6 months will vary in the interval from 0% to 8%.

- A random sample of 20 containers of pesticides from the manufacturer.

<table>
<thead>
<tr>
<th>0.5</th>
<th>3.5</th>
<th>4.4</th>
<th>6.0</th>
<th>6.6</th>
<th>5.4</th>
<th>7.9</th>
<th>4.6</th>
<th>5.4</th>
<th>5.7</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.5</td>
<td>1.1</td>
<td>5.9</td>
<td>2.7</td>
<td>2.3</td>
<td>1.4</td>
<td>1.8</td>
<td>5.8</td>
<td>0.2</td>
<td>7.1</td>
</tr>
</tbody>
</table>
• The manufacturer claimed that the population of potency reductions has a range of 8%. Dividing the range by 4, we obtain an approximate population standard deviation of \( \sigma = 2\% \).

• The approximate null and alternative hypotheses are:

\[
H_0 : \sigma^2 \leq 4 \quad \text{(i.e., the manufacturer's claim is correct.)}
\]

\[
H_a : \sigma^2 > 4 \quad \text{(i.e., there is more variability than claimed by the manufacturer.)}
\]
Answer to Example 7.1 (2)

- Normal probability plot for potency data:

- The variance of the potency data is \( s^2 = 5.45 \).
- The test statistic and rejection region are as follows:

\[
\chi^2 = \frac{(n-1)s^2}{\sigma^2} = \frac{19 \times 5.45}{4} = 25.88 \quad \left[ P(\chi^2_{19} > 25.88) = 0.14 \right]
\]
Example 7.3

- A simulation study was conducted to investigate the effect on the level of the chi-square test of sampling from heavy-tailed and skewed distribution rather than the required normal distribution. The five distributions were normal, uniform (short-tailed), t distribution with df=5 (heavy-tailed), and two gamma distributions, one slightly skewed and other heavily skewed. Some summary statistics about the distributions are given in the following table. Note that each of the distributions has the same variance, $\sigma^2 = 100$, but the skewness and kurtosis of the distributions vary. From each of the distributions, 2500 random samples of sizes 10, 20, and 50 were selected and a test of $H_0: \sigma^2 \leq 100$ were conducted using $\alpha=0.05$ for the hypothesis. A chi-square test of variance was performed for each of the 2500 samples of the various sample sizes from each of the five distributions. What do the results indicate that the sensitivity of the test to sampling from a nonnormal population?

<table>
<thead>
<tr>
<th>Summary Statistics</th>
<th>Normal</th>
<th>Uniform</th>
<th>t (df=5)</th>
<th>Gamma (shape= 1)</th>
<th>Gamma (shape = 0.1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0</td>
<td>17.32</td>
<td>0</td>
<td>10</td>
<td>3.162</td>
</tr>
<tr>
<td>Variance</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>Skewness</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>6.32</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>3</td>
<td>1.8</td>
<td>9</td>
<td>9</td>
<td>63</td>
</tr>
</tbody>
</table>
Answer to Example 7.2

- When the population distribution is symmetric with shorter tails than a normal distribution, the actual probabilities are smaller than 0.05, whereas for a symmetric distribution with heavy tails, the Type I error probabilities are much greater than 0.05. There is strong evidence that the claimed $\alpha$ value of the chi-square test of population variance is very sensitive to nonnormality.
Estimation and Tests for Comparing Two Population Variances

- Application of a test for the equality of two population variances is for evaluating the validity of the equal variance conditions for a two-sample t test.

- When random samples of sizes $n_1$ and $n_2$ have been independently drawn from two normally distributions, the ratio $\frac{s_1^2/\sigma_1^2}{s_2^2/\sigma_2^2} = \frac{s_1^2}{s_2^2} / \frac{\sigma_1^2}{\sigma_2^2}$ possesses a probability distribution in repeated sampling referred to as an $F$ distribution.
Properties of the $F$ Distribution (1)

- $F$ can assume only positive values.

- The $F$ distribution, unlike the normal distribution or the $t$ distribution but like the $\chi^2$ distribution, is nonsymmetrical.

- There are many $F$ distributions, and each one has a different shape. We specify a particular one by designating the degrees of freedom associated with $s_1^2$ and $s_2^2$. We denote these quantities by $d_{f1}$ and $d_{f2}$, respectively.
Statistical Test Comparing $\sigma_1^2$ and $\sigma_2^2$

- A statistical test comparing $\sigma_1^2$ and $\sigma_2^2$ utilizes the test statistic $s_1^2/s_2^2$. When $\sigma_1^2 = \sigma_2^2$, $s_1^2/s_2^2 = 1$ and $s_1^2/s_2^2$ follows an $F$ distribution with $df_1 = n_1 - 1$ and $df_2 = n_2 - 1$.

$$F = \frac{s_1^2}{s_2^2}$$

- The lower-tail values are obtained from the following relationship: Let $F_{\alpha, df_1, df_2}$ be the upper $\alpha$ percentile and $F_{1-\alpha, df_1, df_2}$ be the lower $\alpha$ percentile of an $F$ distribution with $df_1$ and $df_2$.

$$F_{1-\alpha, df_1, df_2} = \frac{1}{F_{\alpha, df_2, df_1}} \quad \frac{S_1^2}{S_2^2} F_L \leq \frac{\sigma_1^2}{\sigma_2^2} \leq \frac{S_1^2}{S_2^2} F_U$$
Example 7.7

- To test hypotheses about the means and standard deviation of HEC and HGMF *E. coli* concentrations.
Answer to Example 7.7 (1)

- Normal probability plots for HGMF and HEC.
Answer to Example 7.7 (2)

- Null hypothesis: \( H_0 : \sigma_1^2 = \sigma_2^2 \) vs. \( H_a : \sigma_1^2 \neq \sigma_2^2 \)
- Summary statistics:

<table>
<thead>
<tr>
<th>Procedure</th>
<th>Sample Size</th>
<th>Mean</th>
<th>Standard Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>HEC</td>
<td>24</td>
<td>7.1346</td>
<td>0.2291</td>
</tr>
<tr>
<td>HGMF</td>
<td>24</td>
<td>6.9529</td>
<td>0.2096</td>
</tr>
</tbody>
</table>

\[
F_0 = \frac{0.2291^2}{0.2096^2} = 1.19 \quad (< F_{0.025,23,23} = 2.31)
\]

- Accept \( H_0 \) and conclude that HEC appears to have a similar degree of variability as HGMF in its determination of \( E. \) coli concentration.
Answer to Example 7.7 (3)

• Both the HEC and HGMF *E. coli* concentration readings appear to be independently random samples from normal populations with a common standard deviation, so we can use a pooled t test to evaluate $H_0 : \mu_1 = \mu_2$ vs. $H_a : \mu_1 \neq \mu_2$

$$|t| = \frac{|\bar{y}_1 - \bar{y}_2|}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = 2.87 > t_{0.025,46} = 2.01$$

• Reject $H_0$ and conclude that there is significant evidence that the average HEC *E. coli* concentration readings different from the average HGMF readings.
Effect on the Level of $F$ test of Sampling from Non-normal Distributions (1)

- A simulation study was conducted to investigate the effect on the level of the $F$ test of sampling from heavy-tailed and skewed distribution rather than the required normal distribution. The five distributions were normal, uniform (short-tailed), $t$ distribution with $df=5$ (heavy-tailed), and two gamma distributions, one slightly skewed and other heavily skewed.

- For each pair of sample sizes $(n_1, n_2) = (10,10), (10,20), \text{ or } (20,20)$, random samples of the specified sizes were selected from one of the five distributions. A test of $H_0: \sigma_1^2 = \sigma_2^2$ vs. $H_a: \sigma_1^2 \neq \sigma_2^2$ was conducted using $F$ test with $\alpha=0.05$. 
Effect on the Level of $F$ test of Sampling from Non-normal Distributions (2)

- Proportion of times $H_0: \sigma_1^2 = \sigma_2^2$ was rejected ($\alpha=0.05$).

<table>
<thead>
<tr>
<th>Sample Sizes</th>
<th>Distribution</th>
<th>Normal</th>
<th>Uniform</th>
<th>t (df=5)</th>
<th>Gamma (shape = 1)</th>
<th>Gamma (shape=0.1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(10,10)</td>
<td>Normal</td>
<td>0.054</td>
<td>0.010</td>
<td>0.121</td>
<td>0.225</td>
<td>0.693</td>
</tr>
<tr>
<td>(10,20)</td>
<td>Normal</td>
<td>0.056</td>
<td>0.0068</td>
<td>0.140</td>
<td>0.236</td>
<td>0.671</td>
</tr>
<tr>
<td>(20,20)</td>
<td>Normal</td>
<td>0.050</td>
<td>0.0044</td>
<td>0.150</td>
<td>0.264</td>
<td>0.673</td>
</tr>
</tbody>
</table>

- When the population distribution is a symmetric short-tailed distribution like the uniform distribution, the value of $\alpha$ is much smaller than the specified value of 0.05. Thus, the probability of Type II errors of the F test would most likely be much larger than what would occur when sampling from normally distributed populations.
Tests for Comparing More Than Two Population Variances

- Hartley $F_{\text{max}}$ test for homogeneity of population variances

\[ H_0: \sigma_1^2 = \sigma_2^2 = \ldots = \sigma_i^2 \text{ homogeneity of variances} \]

\[ H_a: \text{Population variances are not all equal} \]

T.S.: \[ F_{\text{max}} = \frac{s_{\text{max}}^2}{s_{\text{min}}^2} \]

R.R.: For a specified value of $\alpha$, reject $H_0$ if $F_{\text{max}}$ exceeds the tabulated $F$ value (Table 12) for $a = \alpha$, $t$, and $df_2 = n - 1$, where $n$ is the common sample size for the $t$ random samples.

Check assumptions and draw conclusions.
Example 7.8

- It is thought that the temperature can be manipulated to target the power (the strength of the lens) in the manufacture of soft contact lenses. So interest is in comparing the variability in power. The data are coded deviations from target power using monomers from three different suppliers. We wish to test $H_0: \sigma_1^2 = \sigma_2^2 = \sigma_3^2$.
Answer to Example 7.8

- Boxplot of deviation from target power for three suppliers

\[ R.R.: \text{Reject } H_0 \text{ if } F_{\text{max}} \geq F_{\text{max,0.05}} = 6.00 \]

\[ S_{\text{min}}^2 = \min(8.69, 6.89, 80.22) = 6.89 \]

\[ S_{\text{max}}^2 = \min(8.69, 6.89, 80.22) = 80.22 \]

\[ F_{\text{max}} = \frac{S_{\text{max}}^2}{S_{\text{min}}^2} = \frac{80.22}{6.89} = 11.64 > 6.00 \]

- Reject \( H_0 \) and conclude that the variances are not all equal.
An Issue of Hartley $F_{max}$ Test

• The Hartley $F_{max}$ test is quite sensitivity to departures from normality.
  – If the population distributions we are sampling from have a somewhat nonnormal distribution but the variances are equal, the $F_{max}$ will reject $H_0$ and declare the variances to be unequal. The test is detecting the nonnormality of the population distribution, not the unequal variances.

• An alternative approach that does not require the population to have normal distribution is the Levine test. However, the Levine test involves considerably more calculation than the Hartley test. Also, when the populations have a normal distribution, the Hartley test is more powerful than the Levine test.
Levine’s Test for Homogeneity of Population Variances (1)

\[ H_0: \quad \sigma_1^2 = \sigma_2^2 = \ldots = \sigma_i^2 \] homogeneity of variances

\[ H_a: \quad \text{Population variances are not all equal} \]

T.S.: \[ L = \frac{\sum_{i=1}^{t} n_i (\bar{z}_i - \bar{z}_.)^2 \sum_{i=1}^{t} (z_{ij} - \bar{z}_i)^2 /(N - t)}{\sum_{i=1}^{t} \sum_{j=1}^{n_i} (z_{ij} - \bar{z}_i)^2 } \]

R.R.: For a specified value of \( \alpha \), reject \( H_0 \) if \( L \geq F_{\alpha, df_1, df_2} \), where \( df_1 = t - 1 \), \( df_2 = N - t \), \( N = \sum_{i=1}^{t} n_i \), and \( F_{\alpha, df_1, df_2} \) is the upper \( \alpha \) percentile from the \( F \) distribution, Table 8.

Check assumptions and draw conclusions.
Levine's Test for Homogeneity of Population Variances (2)

| Additive | $y_{ij}$ | $\bar{y}_i$ | $z_{ij} = |y_{ij} - 5.80|$ | $\bar{z}_i$ | $(z_{ij} - 4.07)^2$ | $(z_{ij} - 5.06)^2$ |
|----------|---------|-------------|-----------------|---------|----------------|----------------|
| 1        | 4.2     | 5.80        | 1.60            | 4.07    | 6.1009         | 11.9716        |
| 1        | 2.9     | 2.90        | 0.00            |         | 1.3689         | 4.6656         |
| 1        | 0.2     | 5.60        | 5.40            |         | 2.3409         | 0.2916         |
| 1        | 25.7    | 19.90       | 5.80            |         | 250.5889       | 220.2256       |
| 1        | 6.3     | 0.50        | 5.80            |         | 12.7449        | 20.7936        |
| 1        | 7.2     | 1.40        | 5.80            |         | 7.1289         | 13.3956        |
| 1        | 2.3     | 3.50        | 1.20            |         | 0.3249         | 2.4336         |
| 1        | 9.9     | 4.10        | 5.80            |         | 0.0009         | 0.9216         |
| 1        | 5.3     | 0.50        | 5.80            |         | 12.7449        | 20.7936        |
| 1        | 6.5     | 0.70        | 5.80            |         | 11.3569        | 19.0096        |

| Additive | $y_{2j}$ | $\bar{y}_2$ | $z_{2j} = |y_{2j} - 7.55|$ | $\bar{z}_2$ | $(z_{2j} - 8.88)^2$ | $(z_{2j} - 5.06)^2$ |
|----------|---------|-------------|-----------------|---------|----------------|----------------|
| 2        | 0.2     | 7.55        | 7.35            | 8.88    | 2.3409         | 5.2441         |
| 2        | 11.3    | 3.75        | 7.55            |         | 26.3169        | 1.7161         |
| 2        | 0.3     | 7.25        | 7.55            |         | 2.6569         | 4.7961         |
| 2        | 17.1    | 9.55        | 7.55            |         | 0.4489         | 20.1601        |
| 2        | 51.0    | 43.45       | 7.55            |         | 1,195.0849     | 1,473.7921     |
| 2        | 10.1    | 2.55        | 7.55            |         | 40.0689        | 6.3001         |
| 2        | 0.3     | 7.25        | 7.55            |         | 2.6569         | 4.7961         |
| 2        | 0.6     | 6.95        | 7.55            |         | 3.7249         | 3.5721         |