### Basic Differentiation Rules

1. \( \frac{d}{dx}[cu] = cu' \)
2. \( \frac{d}{dx}[u \pm v] = u' \pm v' \)
3. \( \frac{d}{dx}[uv] = uv' + vu' \)
4. \( \frac{d}{dx}\left[\frac{u}{v}\right] = \frac{vu' - uv'}{v^2} \)
5. \( \frac{d}{dx}[c] = 0 \)
6. \( \frac{d}{dx}[u^n] = nu^{n-1}u' \)
7. \( \frac{d}{dx}[x] = 1 \)
8. \( \frac{d}{dx}[\ln u] = \frac{u'}{u} \)
9. \( \frac{d}{dx}[e^u] = e^u u' \)
10. \( \frac{d}{dx}\left[\log_a u\right] = \frac{u'}{(\ln a)u} \)
11. \( \frac{d}{dx}[a^u] = (\ln a)a^u u' \)
12. \( \frac{d}{dx}[\sin u] = (\cos u)u' \)
13. \( \frac{d}{dx}[\cos u] = -(\sin u)u' \)
14. \( \frac{d}{dx}[\tan u] = (\sec^2 u)u' \)
15. \( \frac{d}{dx}[\cot u] = -(\csc^2 u)u' \)
16. \( \frac{d}{dx}[\sec u] = (\sec u \tan u)u' \)
17. \( \frac{d}{dx}[\csc u] = -(\csc u \cot u)u' \)

### Basic Integration Formulas

1. \( \int kf(u) \, du = k \int f(u) \, du \)
2. \( \int [f(u) \pm g(u)] \, du = \int f(u) \, du \pm \int g(u) \, du \)
3. \( \int du = u + C \)
4. \( \int a^u \, du = \left(\frac{1}{\ln a}\right)a^u + C \)
5. \( \int e^u \, du = e^u + C \)
6. \( \int \ln u \, du = u(-1 + \ln u) + C \)
7. \( \int \sin u \, du = -\cos u + C \)
8. \( \int \cos u \, du = \sin u + C \)
9. \( \int \tan u \, du = -\ln|\cos u| + C \)
10. \( \int \cot u \, du = \ln|\sin u| + C \)
11. \( \int \sec u \, du = \ln|\sec u + \tan u| + C \)
12. \( \int \csc u \, du = -\ln|\csc u + \cot u| + C \)
13. \( \int \sec^2 u \, du = \tan u + C \)
14. \( \int \csc^2 u \, du = -\cot u + C \)

### Trigonometric Identities

**Pythagorean Identities**
- \( \sin^2 \theta + \cos^2 \theta = 1 \)
- \( \tan^2 \theta + 1 = \sec^2 \theta \)
- \( \cot^2 \theta + 1 = \csc^2 \theta \)

**Sum or Difference of Two Angles**
- \( \sin(\theta \pm \phi) = \sin \theta \cos \phi \pm \cos \theta \sin \phi \)
- \( \cos(\theta \pm \phi) = \cos \theta \cos \phi \mp \sin \theta \sin \phi \)
- \( \tan(\theta \pm \phi) = \frac{\tan \theta \pm \tan \phi}{1 \mp \tan \theta \tan \phi} \)

**Double Angle**
- \( \sin 2\theta = 2 \sin \theta \cos \theta \)
- \( \cos 2\theta = 2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta \)

**Half Angle**
- \( \sin^2 \frac{\theta}{2} = \frac{1}{2} (1 - \cos \theta) \)
- \( \cos^2 \frac{\theta}{2} = \frac{1}{2} (1 + \cos \theta) \)

**Reduction Formulas**
- \( \sin(-\theta) = -\sin \theta \)
- \( \cos(-\theta) = \cos \theta \)
- \( \tan(-\theta) = -\tan \theta \)

**Sum or Difference of Two Angles**
- \( \sin(\theta \pm \phi) = \sin \theta \cos \phi \pm \cos \theta \sin \phi \)
- \( \cos(\theta \pm \phi) = \cos \theta \cos \phi \mp \sin \theta \sin \phi \)
- \( \tan(\theta \pm \phi) = \frac{\tan \theta \pm \tan \phi}{1 \mp \tan \theta \tan \phi} \)
Section 11.3

Partial Derivatives

- Find and use partial derivatives of a function of two variables.
- Find and use partial derivatives of a function of three or more variables.
- Find higher-order partial derivatives of a function of two or three variables.

Partial Derivatives of a Function of Two Variables

In applications of functions of several variables, the question often arises, "How will the value of a function be affected by a change in one of its independent variables?" You can answer this by considering the independent variables one at a time. For example, to determine the effect of a catalyst in an experiment, a chemist could conduct the experiment several times using varying amounts of the catalyst, while keeping constant other variables such as temperature and pressure. You can use a similar procedure to determine the rate of change of a function $f$ with respect to one of its several independent variables. This process is called partial differentiation, and the result is referred to as the partial derivative of $f$ with respect to the chosen independent variable.

Definition of Partial Derivatives of a Function of Two Variables

If $z = f(x, y)$, then the first partial derivatives of $f$ with respect to $x$ and $y$ are the functions $f_x$ and $f_y$ defined by

$$ f_x(x, y) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} $$

$$ f_y(x, y) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} $$

provided the limits exist.

This definition indicates that if $z = f(x, y)$, then to find $f_x$, you consider $y$ constant and differentiate with respect to $x$. Similarly, to find $f_y$, you consider $x$ constant and differentiate with respect to $y$.

**Example 1** Finding Partial Derivatives

Find the partial derivatives $f_x$ and $f_y$ for the function

$$ f(x, y) = 3x - x^3y^2 + 2x^3y. $$

**Solution** Considering $y$ to be constant and differentiating with respect to $x$ produces

$$ f(x, y) = 3x - x^3y^2 + 2x^3y. $$

Partial derivative with respect to $x$

Considering $x$ to be constant and differentiating with respect to $y$ produces

$$ f(x, y) = 3x - x^3y^2 + 2x^3y. $$

Partial derivative with respect to $y$
Notation for First Partial Derivatives

For \( z = f(x, y) \), the partial derivatives \( f_x \) and \( f_y \) are denoted by

\[
\frac{\partial}{\partial x} f(x, y) = f_x(x, y) = z_x = \frac{\partial z}{\partial x}
\]

and

\[
\frac{\partial}{\partial y} f(x, y) = f_y(x, y) = z_y = \frac{\partial z}{\partial y}
\]

The first partials evaluated at the point \((a, b)\) are denoted by

\[
\frac{\partial z}{\partial x}\bigg|_{(a, b)} = f_x(a, b) \quad \text{and} \quad \frac{\partial z}{\partial y}\bigg|_{(a, b)} = f_y(a, b).
\]

**EXAMPLE 2** Finding and Evaluating Partial Derivatives

For \( f(x, y) = xe^{x+y} \), find \( f_x \) and \( f_y \), and evaluate each at the point \((1, \ln 2)\).

**Solution**

Because

\[
f_x(x, y) = xe^{x+y}(2y) = 2xye^{x+y}
\]

the partial derivative of \( f \) with respect to \( x \) at \((1, \ln 2)\) is

\[
f_x(1, \ln 2) = 2e^{1+\ln 2} = 4 \ln 2 + 2.
\]

Because

\[
f_y(x, y) = xe^{x+y}(x^2) = xe^{x+y}
\]

the partial derivative of \( f \) with respect to \( y \) at \((1, \ln 2)\) is

\[
f_y(1, \ln 2) = e^{1+\ln 2} = 2.
\]

The partial derivatives of a function of two variables, \( z = f(x, y) \), have a useful geometric interpretation. If \( y = y_0 \) then \( z = f(x, y_0) \) represents the curve formed by intersecting the surface \( z = f(x, y) \) with the plane \( y = y_0 \) as shown in Figure 11.29. Therefore,

\[
f_x(x_0, y_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}
\]

represents the slope of this curve at the point \((x_0, y_0, f(x_0, y_0))\). Note that both the curve and the tangent line lie in the plane \( y = y_0 \). Similarly,

\[
f_y(x_0, y_0) = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}
\]

represents the slope of the curve given by the intersection of \( z = f(x, y) \) and the plane \( x = x_0 \) at \((x_0, y_0, f(x_0, y_0))\), as shown in Figure 11.30.

Informally, the values of \( df/\partial x \) and \( df/\partial y \) at the point \((x_0, y_0, z_0)\) denote the slopes of the surface in the \( x \)- and \( y \)-directions, respectively.
**EXAMPLE 3** Finding the Slopes of a Surface in the x- and y-Directions

Find the slopes in the x-direction and in the y-direction of the surface given by

\[ f(x, y) = \frac{-x^2}{2} - y^2 + \frac{25}{8} \]

at the point \( \left( \frac{1}{2}, 1, 2 \right) \).

**Solution** The partial derivatives of \( f \) with respect to \( x \) and \( y \) are

\[ f_x(x, y) = -x \quad \text{and} \quad f_y(x, y) = -2y. \]

So, in the x-direction, the slope is

\[ f_x\left( \frac{1}{2}, 1 \right) = -\frac{1}{2} \]

and in the y-direction, the slope is

\[ f_y\left( \frac{1}{2}, 1 \right) = -2. \]

**EXAMPLE 4** Finding the Slopes of a Surface in the x- and y-Directions

Find the slopes of the surface given by

\[ f(x, y) = 1 - (x - 1)^2 - (y - 2)^2 \]

at the point \((1, 2, 1)\) in the x-direction and in the y-direction.

**Solution** The partial derivatives of \( f \) with respect to \( x \) and \( y \) are

\[ f_x(x, y) = -2(x - 1) \quad \text{and} \quad f_y(x, y) = -2(y - 2). \]

So, at the point \((1, 2, 1)\), the slopes in the x- and y-directions are

\[ f_x(1, 2) = -2(1 - 1) = 0 \quad \text{and} \quad f_y(1, 2) = -2(2 - 2) = 0 \]

as shown in Figure 11.32.
No matter how many variables are involved, partial derivatives can be interpreted as *rates of change*.

**Example 5**  
Using Partial Derivatives to Find Rates of Change

The area of a parallelogram with adjacent sides $a$ and $b$ and included angle $\theta$ is given by $A = ab \sin \theta$, as shown in Figure 11.33.

**a.** Find the rate of change of $A$ with respect to $a$ for $a = 10$, $b = 20$, and $\theta = \frac{\pi}{6}$.

**b.** Find the rate of change of $A$ with respect to $\theta$ for $a = 10$, $b = 20$, and $\theta = \frac{\pi}{6}$.

**Solution**

a. To find the rate of change of the area with respect to $a$, hold $b$ and $\theta$ constant and differentiate with respect to $a$ to obtain

$$\frac{\partial A}{\partial a} = b \sin \theta$$

Find partial with respect to $a$.

$$\frac{\partial A}{\partial a} = 20 \sin \frac{\pi}{6} = 10.$$ 

Substitute for $b$ and $\theta$.

b. To find the rate of change of the area with respect to $\theta$, hold $a$ and $b$ constant and differentiate with respect to $\theta$ to obtain

$$\frac{\partial A}{\partial \theta} = ab \cos \theta$$

Find partial with respect to $\theta$.

$$\frac{\partial A}{\partial \theta} = 200 \cos \frac{\pi}{6} = 100\sqrt{3}.$$ 

Substitute for $a$, $b$, and $\theta$.

**Partial Derivatives of a Function of Three or More Variables**

The concept of a partial derivative can be extended naturally to functions of three or more variables. For instance, if $w = f(x, y, z)$, there are three partial derivatives, each of which is formed by holding two of the variables constant. That is, to define the partial derivative of $w$ with respect to $x$, consider $y$ and $z$ to be constant and differentiate with respect to $x$. A similar process is used to find the derivatives of $w$ with respect to $y$ and with respect to $z$.

$$\frac{\partial w}{\partial x} = f_x(x, y, z) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y, z) - f(x, y, z)}{\Delta x}$$

$$\frac{\partial w}{\partial y} = f_y(x, y, z) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y, z) - f(x, y, z)}{\Delta y}$$

$$\frac{\partial w}{\partial z} = f_z(x, y, z) = \lim_{\Delta z \to 0} \frac{f(x, y, z + \Delta z) - f(x, y, z)}{\Delta z}$$

In general, if $w = f(x_1, x_2, \ldots, x_n)$, there are $n$ partial derivatives denoted by

$$\frac{\partial w}{\partial x_k} = f_{x_k}(x_1, x_2, \ldots, x_n), \quad k = 1, 2, \ldots, n.$$ 

To find the partial derivative with respect to one of the variables, hold the other variables constant and differentiate with respect to the given variable.
EXAMPLE 6  Finding Partial Derivatives

a. To find the partial derivative of \( f(x, y, z) = xy + yz^2 + xz \) with respect to \( z \), consider \( x \) and \( y \) to be constant and obtain
\[
\frac{\partial}{\partial z}[xy + yz^2 + xz] = 2yz + x.
\]

b. To find the partial derivative of \( f(x, y, z) = z \sin(xy^2 + 2z) \) with respect to \( z \), consider \( x \) and \( y \) to be constant. Then, using the Product Rule, you obtain
\[
\frac{\partial}{\partial z}[z \sin(xy^2 + 2z)] = (z) \frac{\partial}{\partial z}[\sin(xy^2 + 2z)] + z \cos(xy^2 + 2z) \frac{\partial}{\partial z}[2z]
\]
\[
= (z) \cos(xy^2 + 2z) \frac{2y}{z} + \sin(xy^2 + 2z) \cdot 2z
\]
\[
= 2z \cos(xy^2 + 2z) + \sin(xy^2 + 2z).
\]

c. To find the partial derivative of \( f(x, y, z, w) = (x + y + z)/w \) with respect to \( w \), consider \( x, y, \) and \( z \) to be constant and obtain
\[
\frac{\partial}{\partial w} \left[ \frac{x + y + z}{w} \right] = -\frac{x + y + z}{w^2}.
\]

Higher-Order Partial Derivatives

As is true for ordinary derivatives, it is possible to take second, third, and higher partial derivatives of a function of several variables, provided such derivatives exist. Higher-order derivatives are denoted by the order in which the differentiation occurs. For instance, the function \( z = f(x, y) \) has the following second partial derivatives.

1. Differentiate twice with respect to \( x \):
\[
\frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}.
\]

2. Differentiate twice with respect to \( y \):
\[
\frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}.
\]

3. Differentiate first with respect to \( x \) and then with respect to \( y \):
\[
\frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy}.
\]

4. Differentiate first with respect to \( y \) and then with respect to \( x \):
\[
\frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx}.
\]

The third and fourth cases are called mixed partial derivatives.
EXAMPLE 7 Finding Second Partial Derivatives

Find the second partial derivatives of \( f(x, y) = 3xy^2 - 2y + 5x^2y^2 \), and determine the value of \( f_{xy}(-1, 2) \).

Solution Begin by finding the first partial derivatives with respect to \( x \) and \( y \).

\[
\begin{align*}
    f_x(x, y) &= 3y^2 + 10xy^2 \\
    f_y(x, y) &= 6xy - 2 + 10x^2y
\end{align*}
\]

Then, differentiate each of these with respect to \( x \) and \( y \).

\[
\begin{align*}
    f_{xx}(x, y) &= 10y^2 \\
    f_{xy}(x, y) &= 6x + 10x^2y \\
    f_{yx}(x, y) &= 6y + 20xy \\
    f_{yy}(x, y) &= 6x + 20xy
\end{align*}
\]

At \((-1, 2)\), the value of \( f_{xy} \) is \( f_{xy}(-1, 2) = 12 - 40 = -28 \).

NOTE Notice in Example 7 that the two mixed partials are equal. Sufficient conditions for this occurrence are given in Theorem 11.3.

THEOREM 11.3 Equality of Mixed Partial Derivatives

If \( f \) is a function of \( x \) and \( y \) such that \( f_{xy} \) and \( f_{yx} \) are continuous on an open disk \( R \), then, for every \((x, y)\) in \( R \),

\[
f_{xy}(x, y) = f_{yx}(x, y).
\]

Theorem 11.3 also applies to a function \( f \) of three or more variables as long as all second partial derivatives are continuous. For example, if \( w = f(x, y, z) \) and all the second partial derivatives are continuous in an open region \( R \), then at each point in \( R \) the order of differentiation of the mixed second partial derivatives is irrelevant. If the third partial derivatives of \( f \) are also continuous, the order of differentiation of the mixed third partial derivatives is irrelevant.

EXAMPLE 8 Finding Higher-Order Partial Derivatives

Show that \( f_{xz} = f_{zx} \) and \( f_{xz} = f_{zx} = f_{zxx} \) for the function given by

\[
f(x, y, z) = ye^x + x \ln z,
\]

Solution

First partials:

\[
\begin{align*}
    f_x(x, y, z) &= ye^x + \ln z \\
    f_y(x, y, z) &= e^x \\
    f_z(x, y, z) &= \frac{x}{z}
\end{align*}
\]

Second partials (note that the first two are equal):

\[
\begin{align*}
    f_{xx}(x, y, z) &= \frac{1}{z} \\
    f_{xy}(x, y, z) &= \frac{1}{z} \\
    f_{xz}(x, y, z) &= -\frac{x}{z^2}
\end{align*}
\]

Third partials (note that all three are equal):

\[
\begin{align*}
    f_{xxx}(x, y, z) &= -\frac{1}{z^2} \\
    f_{xxy}(x, y, z) &= -\frac{1}{z^2} \\
    f_{xxz}(x, y, z) &= -\frac{1}{z^2}
\end{align*}
\]
Section 11.4

Differentials and the Chain Rule

- Understand the concepts of increments and differentials.
- Extend the concept of differentiability to a function of two variables.
- Use a differential as an approximation.
- Use the Chain Rules for functions of several variables.
- Find partial derivatives implicitly.

Increments and Differentials

The concepts of increments and differentials can be generalized to functions of two or more variables. Recall from Section 3.7 that for \( y = f(x) \), the differential of \( y \) was defined as \( dy = f'(x) \, dx \). Similar terminology is used for a function of two variables, \( z = f(x, y) \). That is, \( \Delta x \) and \( \Delta y \) are the increments of \( x \) and \( y \), and the increment of \( z \) is given by

\[
\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y).
\]

Increment of \( z \)

Definition of Total Differential

If \( z = f(x, y) \) and \( \Delta x \) and \( \Delta y \) are increments of \( x \) and \( y \), then the differentials of the independent variables \( x \) and \( y \) are

\[
dx = \Delta x \quad \text{and} \quad dy = \Delta y
\]

and the total differential of the dependent variable \( z \) is

\[
dz = \frac{\partial z}{\partial x} \, dx + \frac{\partial z}{\partial y} \, dy = f_x(x, y) \, dx + f_y(x, y) \, dy.
\]

This definition can be extended to a function of three or more variables. For instance, if \( w = f(x, y, z, u) \), then \( dx = \Delta x \), \( dy = \Delta y \), \( dz = \Delta z \), \( du = \Delta u \), and the total differential of \( w \) is

\[
dw = \frac{\partial w}{\partial x} \, dx + \frac{\partial w}{\partial y} \, dy + \frac{\partial w}{\partial z} \, dz + \frac{\partial w}{\partial u} \, du.
\]

EXAMPLE 1 Finding the Total Differential

Find the total differential for each function.

a. \( z = 2x \sin y - 3x^2y^2 \)

b. \( w = x^2 + y^2 + z^2 \)

Solution

a. The total differential \( dz \) for \( z = 2x \sin y - 3x^2y^2 \) is

\[
dz = \frac{\partial z}{\partial x} \, dx + \frac{\partial z}{\partial y} \, dy
\]

\[= (2 \sin y - 6xy^2) \, dx + (2x \cos y - 6x^2y) \, dy.
\]

b. The total differential \( dw \) for \( w = x^2 + y^2 + z^2 \) is

\[
dw = \frac{\partial w}{\partial x} \, dx + \frac{\partial w}{\partial y} \, dy + \frac{\partial w}{\partial z} \, dz
\]

\[= 2x \, dx + 2y \, dy + 2z \, dz.
\]
Differentiability

In Section 3.7, you learned that for a differentiable function given by \( y = f(x) \), you can use the differential \( dy = f'(x) \, dx \) as an approximation (for small \( dx \)) to the value \( \Delta y = f(x + \Delta x) - f(x) \). When a similar approximation is possible for a function of two variables, the function is said to be differentiable. This is stated explicitly in the following definition.

**Definition of Differentiability**

A function \( f \) given by \( z = f(x, y) \) is differentiable at \((x_0, y_0)\) if \( \Delta z \) can be written in the form

\[
\Delta z = f_x(x_0, y_0) \Delta x + f_y(x_0, y_0) \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y
\]

where both \( \varepsilon_1 \) and \( \varepsilon_2 \to 0 \) as \((\Delta x, \Delta y)\to(0, 0)\). The function \( f \) is differentiable in a region \( R \) if it is differentiable at each point in \( R \).

**EXAMPLE 2**  Showing That a Function Is Differentiable

Show that the function given by

\[
f(x, y) = x^2 + 3y
\]

is differentiable at every point in the plane.

**Solution**  Letting \( z = f(x, y) \), the increment of \( z \) at an arbitrary point \((x, y)\) in the plane is

\[
\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)
\]

Increment of \( z \)

\[
= (x^2 + 2x\Delta x + \Delta^2 x + 3(y + \Delta y) - (x^2 + 3y)
\]

\[
= 2x\Delta x + \Delta^2 x + 3\Delta y
\]

\[
= 2x\Delta x + 3\Delta y + \Delta x(\Delta y) + 0(\Delta y)
\]

\[
= f_x(x, y) \Delta x + f_y(x, y) \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y
\]

where \( \varepsilon_1 = \Delta x \) and \( \varepsilon_2 = 0 \). Because \( \varepsilon_1 \to 0 \) and \( \varepsilon_2 \to 0 \) as \((\Delta x, \Delta y)\to(0, 0)\), it follows that \( f \) is differentiable at every point in the plane. The graph of \( f \) is shown in Figure 11.34.

Be sure you see that the term "differentiable" is used differently for functions of two variables than for functions of one variable. A function of one variable is differentiable at a point if its derivative exists at the point. However, for a function of two variables, the existence of the partial derivatives \( f_x \) and \( f_y \) does not guarantee that the function is differentiable (see Exercises 87 and 88). The following theorem gives a sufficient condition for differentiability of a function of two variables. A proof of Theorem 11.4 is given in Appendix A.

**THEOREM 11.4**  Sufficient Condition for Differentiability

If \( f \) is a function of \( x \) and \( y \), where \( f_x \) and \( f_y \) are continuous in an open region \( R \), then \( f \) is differentiable on \( R \).
Approximation by Differentials

Theorem 11.4 tells you that you can choose \((x + \Delta x, y + \Delta y)\) close enough to \((x, y)\) to make \(e_x \Delta x\) and \(e_y \Delta y\) insignificant. In other words, for small \(\Delta x\) and \(\Delta y\), you can use the approximation

\[
\Delta z \approx dz.
\]

This approximation is illustrated graphically in Figure 11.35. Recall that the partial derivatives \(\frac{\partial z}{\partial x}\) and \(\frac{\partial z}{\partial y}\) can be interpreted as the slopes of the surface in the \(x\) and \(y\)-directions. This means that

\[
dz = \frac{\partial z}{\partial x} \Delta x + \frac{\partial z}{\partial y} \Delta y
\]

represents the change in height of a plane that is tangent to the surface at the point \((x, y, f(x, y))\). Because a plane in space is represented by a linear equation in the variables \(x\), \(y\), and \(z\), the approximation of \(\Delta z\) by \(dz\) is called a linear approximation. You will learn more about this geometric interpretation in Section 11.6.

**Example 3 Using a Differential as an Approximation**

Use the differential \(dz\) to approximate the change in \(z = \sqrt{4 - x^2 - y^2}\) as \((x, y)\) moves from the point \((1, 1)\) to the point \((1.01, 0.97)\). Compare this approximation with the exact change in \(z\).

**Solution** Letting \((x, y) = (1, 1)\) and \((x + \Delta x, y + \Delta y) = (1.01, 0.97)\) produces \(dx = \Delta x = 0.01\) and \(dy = \Delta y = -0.03\). So, the change in \(z\) can be approximated by

\[
\Delta z \approx dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = \frac{-x}{\sqrt{4 - x^2 - y^2}} \Delta x + \frac{-y}{\sqrt{4 - x^2 - y^2}} \Delta y.
\]

When \(x = 1\) and \(y = 1\), you have

\[
\Delta z \approx \frac{-1}{\sqrt{2}} (0.01) - \frac{1}{\sqrt{2}} (-0.03) = 0.02\sqrt{2} = 0.02 (0.01) = 0.0141.
\]

In Figure 11.36 you can see that the exact change corresponds to the difference in the heights of two points on the surface of a hemisphere. This difference is given by

\[
\Delta z = f(1.01, 0.97) - f(1, 1) = \sqrt{4 - (1.01)^2 - (0.97)^2} - \sqrt{4 - 1^2 - 1^2} = 0.0137.
\]

A function of three variables \(w = f(x, y, z)\) is called **differentiable** at \((x, y, z)\) if provided that

\[
\Delta w = f(x + \Delta x, y + \Delta y, z + \Delta z) - f(x, y, z)
\]

can be written in the form

\[
\Delta w = f_1 \Delta x + f_2 \Delta y + f_3 \Delta z + e_1 \Delta x + e_2 \Delta y + e_3 \Delta z
\]

where \(e_1\), \(e_2\), and \(e_3\) tend to zero as \((\Delta x, \Delta y, \Delta z) \to (0, 0, 0)\). With this definition of differentiability, Theorem 11.4 has the following extension for functions of three variables: If \(f\) is a function of \(x\), \(y\), and \(z\), where \(f, f_1, f_2,\) and \(f_3\) are continuous in an open region \(R\), then \(f\) is differentiable on \(R\).

In Section 11.7, you used differentials to approximate the propagated error introduced by an error in measurement. This application of differentials is further illustrated in Example 4.
EXAMPLE 4  Error Analysis

The possible error involved in measuring each dimension of a rectangular box is ±0.1 millimeter. The dimensions of the box are \( x = 50 \) centimeters, \( y = 20 \) centimeters, and \( z = 15 \) centimeters, as shown in Figure 11.37. Use \( dV \) to estimate the propagated error and the relative error in the calculated volume of the box.

Solution  The volume of the box is given by \( V = xyz \), and so

\[
dV = \frac{\partial V}{\partial x} \, dx + \frac{\partial V}{\partial y} \, dy + \frac{\partial V}{\partial z} \, dz = yz \, dx + xz \, dy + xy \, dz.
\]

Using 0.1 millimeter = 0.01 centimeter, you have \( dx = dy = dz = \pm 0.01 \), and the propagated error is approximately

\[
dV = (20)(15)(\pm 0.01) + (50)(15)(\pm 0.01) + (50)(20)(\pm 0.01)
\]

\[
= 300(\pm 0.01) + 750(\pm 0.01) + 1000(\pm 0.01)
\]

\[
= 2050(\pm 0.01) = \pm 20.5 \text{ cubic centimeters}.
\]

Because the measured volume is

\[
V = (50)(20)(15) = 15,000 \text{ cubic centimeters}
\]

the relative error, \( \Delta V / V \), is approximately

\[
\frac{\Delta V}{V} = \frac{dV}{V} = \frac{20.5}{15,000} \approx 0.14\% .
\]

As is true for a function of a single variable, if a function in two or more variables is differentiable at a point, it is also continuous there.

THEOREM 11.5  Differentiability Implies Continuity

If a function of \( x \) and \( y \) is differentiable at \((x_0, y_0)\), then it is continuous at \((x_0, y_0)\).

Proof  Let \( f \) be differentiable at \((x_0, y_0)\), where \( z = f(x, y) \). Then

\[
\Delta z = [f_x(x_0, y_0) + e_1] \Delta x + [f_y(x_0, y_0) + e_2] \Delta y
\]

where both \( e_1 \) and \( e_2 \to 0 \) as \( \Delta x, \Delta y \to (0, 0) \). However, by definition, you know that \( \Delta z \) is given by

\[
\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0).
\]

Letting \( x = x_0 + \Delta x \) and \( y = y_0 + \Delta y \) produces

\[
f(x, y) - f(x_0, y_0) = [f_x(x_0, y_0) + e_1] \Delta x + [f_y(x_0, y_0) + e_2] \Delta y
\]

\[
= [f_x(x_0, y_0) + e_1](x - x_0) + [f_y(x_0, y_0) + e_2](y - y_0).
\]

Taking the limit as \((x, y) \to (x_0, y_0)\), you have

\[
\lim_{(x, y) \to (x_0, y_0)} f(x, y) = f(x_0, y_0)
\]

which means that \( f \) is continuous at \((x_0, y_0)\).

Remember that the existence of \( f_x \) and \( f_y \) is not sufficient to guarantee differentiability (see Exercises 87 and 88).
Chain Rules for Functions of Several Variables

Your work with differentials provides the basis for the extension of the Chain Rule to functions of two variables. There are two cases—the first case involves $w$ as a function of $x$ and $y$, where $x$ and $y$ are functions of $t$. This diagram represents the derivative of $w$ with respect to $t$. Figure 11.38

**THEOREM 11.6 Chain Rule: One Independent Variable**

Let $w = f(x, y)$, where $f$ is a differentiable function of $x$ and $y$. If $x = g(t)$ and $y = h(t)$, where $g$ and $h$ are differentiable functions of $t$, then $w$ is a differentiable function of $t$, and

$$rac{dw}{dt} = rac{∂w}{∂x} \frac{dx}{dt} + \frac{∂w}{∂y} \frac{dy}{dt}$$

See Figure 11.38.

**EXAMPLE 5 Using the Chain Rule with One Independent Variable**

Let $w = x^2y - y^2$, where $x = \sin t$ and $y = e^t$. Find $dw/dt$ when $t = 0$.

**Solution** By the Chain Rule for one independent variable, you have

$$\frac{dw}{dt} = \frac{∂w}{∂x} \frac{dx}{dt} + \frac{∂w}{∂y} \frac{dy}{dt}$$

$$= 2xycos t + (x^2 - 2y)e^t$$

$$= 2(\sin t)(e^t)(\cos t) + (\sin^2 t - 2e^t)e^t$$

$$= 2e^t \sin t \cos t + e^t \sin^2 t - 2e^t.$$

When $t = 0$, it follows that

$$\frac{dw}{dt} = -2.$$

The Chain Rules presented in this section provide alternative techniques for solving many problems in single-variable calculus. For instance, in Example 5, you could have used single-variable techniques to find $dw/dt$ by first writing $w$ as a function of $t$,

$$w = x^2y - y^2$$

$$= (\sin t)^2(e^t) - (e^t)^2$$

$$= e^t \sin^2 t - e^{2t}$$

and then differentiating as usual.

$$\frac{dw}{dt} = 2e^t \sin t \cos t + e^t \sin^2 t - 2e^t.$$

The Chain Rule in Theorem 11.6 can be extended to any number of variables. For example, if each $x_i$ is a differentiable function of a single variable $t$, then for

$$w = f(x_1, x_2, \ldots, x_n)$$

you have

$$\frac{dw}{dt} = \frac{∂w}{∂x_1} \frac{dx_1}{dt} + \frac{∂w}{∂x_2} \frac{dx_2}{dt} + \ldots + \frac{∂w}{∂x_n} \frac{dx_n}{dt}.$$
**EXAMPLE 6** An Application of a Chain Rule to Related Rates

Two objects are traveling in elliptical paths given by the following parametric equations.

\[ x_1 = 4 \cos t \quad \text{and} \quad y_1 = 2 \sin t \quad \text{First object} \]
\[ x_2 = 2 \sin 2t \quad \text{and} \quad y_2 = 3 \cos 2t \quad \text{Second object} \]

At what rate is the distance between the two objects changing when \( t = \pi \)?

**Solution** From Figure 11.39, you can see that the distance \( s \) between the two objects is given by

\[ s = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \]

and that when \( t = \pi \), you have \( x_1 = -4, y_1 = 0, x_2 = 0, y_2 = 3 \), and

\[ s = \sqrt{(0 - 4)^2 + (3 - 0)^2} = 5. \]

When \( t = \pi \), the partial derivatives of \( s \) are as follows.

\[ \frac{\partial s}{\partial x_1} = \frac{-(x_2 - x_1)}{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \frac{-4}{(0 - 4)^2 + (3 - 0)^2} = \frac{-4}{5} \]
\[ \frac{\partial s}{\partial y_1} = \frac{-(y_2 - y_1)}{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \frac{-3}{(0 - 4)^2 + (3 - 0)^2} = \frac{-3}{5} \]
\[ \frac{\partial s}{\partial x_2} = \frac{(y_2 - y_1)}{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \frac{4}{(0 + 4)^2 + (3 - 0)^2} = \frac{4}{5} \]
\[ \frac{\partial s}{\partial y_2} = \frac{(y_2 - y_1)}{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \frac{3}{(0 + 4)^2 + (3 - 0)^2} = \frac{3}{5} \]

When \( t = \pi \), the derivatives of \( x_1, y_1, x_2, \) and \( y_2 \) are

\[ \frac{dx_1}{dt} = -4 \sin t = 0 \quad \frac{dy_1}{dt} = 2 \cos t = -2 \]
\[ \frac{dx_2}{dt} = 4 \cos 2t = 4 \quad \frac{dy_2}{dt} = -6 \sin 2t = 0. \]

So, using the appropriate Chain Rule, you know that the distance is changing at a rate of

\[ \frac{ds}{dt} = \frac{\partial s}{\partial x_1} \frac{dx_1}{dt} + \frac{\partial s}{\partial y_1} \frac{dy_1}{dt} + \frac{\partial s}{\partial x_2} \frac{dx_2}{dt} + \frac{\partial s}{\partial y_2} \frac{dy_2}{dt} \]
\[ = \left( -\frac{4}{5} \right)(0) + \left( -\frac{3}{5} \right)(-2) + \left( \frac{4}{5} \right)(4) + \left( \frac{3}{5} \right)(0) \]
\[ = \frac{22}{5}. \]

In Example 6, note that \( s \) is the function of four intermediate variables, \( x_1, y_1, x_2, \) and \( y_2 \), each of which is a function of a single variable \( t \). Another type of composite function is one in which the intermediate variables are themselves functions of more than one variable. For instance, if \( w = f(x, y) \), where \( x = g(s, t) \) and \( y = h(s, t) \), it follows that \( w \) is a function of \( s \) and \( t \), and you can consider the partial derivatives of \( w \) with respect to \( s \) and \( t \). One way to find these partial derivatives is to write \( w \) as a function of \( s \) and \( t \) explicitly by substituting the equations \( x = g(s, t) \) and \( y = h(s, t) \) into the equation \( w = f(x, y) \). Then you can find the partial derivatives in the usual way, as demonstrated in the next example.
**Example 7**  Finding Partial Derivatives by Substitution

Find \( \frac{\partial w}{\partial s} \) and \( \frac{\partial w}{\partial t} \) for \( w = 2xy \), where \( x = s^2 + t^2 \) and \( y = s/t \).

**Solution**  Begin by substituting \( x = s^2 + t^2 \) and \( y = s/t \) into the equation \( w = 2xy \) to obtain

\[
 w = 2xy = 2(s^2 + t^2)
\]

Then, to find \( \frac{\partial w}{\partial s} \), hold \( t \) constant and differentiate with respect to \( s \).

\[
 \frac{\partial w}{\partial s} = \frac{2(2s^2 + 2t^2)}{t} = 4s^2 + 2t^2
\]

Similarly, to find \( \frac{\partial w}{\partial t} \), hold \( s \) constant and differentiate with respect to \( t \) to obtain

\[
 \frac{\partial w}{\partial t} = 2\left(-\frac{s^3 + st^2}{t^2}\right) = \frac{2st^2 - 2s^3}{t^2}.
\]

Theorem 11.7 gives an alternative method for finding the partial derivatives in Example 7, without explicitly writing \( w \) as a function of \( s \) and \( t \).

**Theorem 11.7**  Chain Rule: Two Independent Variables

Let \( w = f(x, y) \), where \( f \) is a differentiable function of \( x \) and \( y \). If \( x = g(s, t) \) and \( y = h(s, t) \) such that the first partials \( \frac{\partial w}{\partial s}, \frac{\partial w}{\partial t}, \frac{\partial x}{\partial s}, \frac{\partial y}{\partial s}, \text{ and } \frac{\partial y}{\partial t} \) all exist, then \( \frac{\partial w}{\partial s} \) and \( \frac{\partial w}{\partial t} \) exist and are given by

\[
 \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial t}.
\]

**Proof**  To obtain \( \frac{\partial w}{\partial s} \), hold \( t \) constant and apply Theorem 11.6 to obtain the desired result. Similarly, for \( \frac{\partial w}{\partial t} \), hold \( s \) constant and apply Theorem 11.6.

**Note**  The Chain Rule in this theorem is shown schematically in Figure 11.40.
**Example 8** The Chain Rule with Two Independent Variables

Use the Chain Rule to find \( \frac{\partial w}{\partial s} \) and \( \frac{\partial w}{\partial t} \) for \( w = 2xy \) where \( x = s^2 + t^2 \) and \( y = \frac{1}{t} \).

**Solution** Note that these same partials were found in Example 7. This time, using Theorem 11.7, you can hold \( t \) constant and differentiate with respect to \( s \) to obtain

\[
\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s}
= 2y(2s) + 2x\left(\frac{1}{t}\right)
= 2\left(\frac{1}{t}\right)(2s) + 2(s^2 + t^2)\left(\frac{1}{t}\right)
= \frac{4s^2}{t} + \frac{2s^2 + 2t^2}{t} = \frac{6s^2 + 2t^2}{t}.
\]

Similarly, holding \( s \) constant gives

\[
\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial t}
= 2y(2t) + 2x\left(\frac{1}{t^2}\right)
= 2\left(\frac{1}{t^2}\right)(2t) + 2(s^2 + t^2)\left(\frac{-s}{t^2}\right)
= \frac{4st^2}{t^2} - \frac{2x^3 - 2st^2}{t^2} = \frac{2x^2 - 2s^2}{t^2}.
\]

**Example 9** The Chain Rule for a Function of Three Variables

Find \( \frac{\partial w}{\partial s} \) and \( \frac{\partial w}{\partial t} \) when \( s = 1 \) and \( t = 2\pi \) for the function given by

\[ w = xy + yz + xz \]

where \( x = s \cos t \), \( y = s \sin t \), and \( z = t \).

**Solution** By extending the result of Theorem 11.7, you have

\[
\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial s}
= (y + z)(\cos t) + (x + z)(\sin t) + (y + x)(0)
= (y + z)(\cos t) + (x + z)(\sin t).
\]

When \( s = 1 \) and \( t = 2\pi \), you have \( x = 1 \), \( y = 0 \), and \( z = 2\pi \). So, \( \frac{\partial w}{\partial s} = (0 + 2\pi)(1) + (1 + 2\pi)(0) = 2\pi \). Furthermore,

\[
\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial t}
= (y + z)(-s \sin t) + (x + z)(s \cos t) + (y + x)(1)
\]

and for \( s = 1 \) and \( t = 2\pi \), it follows that

\[
\frac{\partial w}{\partial t} = (0 + 2\pi)(0) + (1 + 2\pi)(1) + (0 + 1)(1) = 2 + 2\pi.
\]
Implicit Partial Differentiation

This section concludes with an application of the Chain Rule to determine the derivative of a function defined implicitly. Suppose that $x$ and $y$ are related by the equation $F(x, y) = 0$, where it is assumed that $y = f(x)$ is a differentiable function of $x$. To find $dy/dx$, you could use the techniques discussed in Section 2.5. However, you will see that the Chain Rule provides a convenient alternative. If you consider the function given by

$$w = F(x, y) = F(x, f(x))$$

you can apply Theorem 11.6 to obtain

$$\frac{dw}{dx} = F_x(x, y) \frac{dx}{dx} + F_y(x, y) \frac{dy}{dx}.$$  

Because $w = F(x, y) = 0$ for all $x$ in the domain of $f$, you know that $dw/dx = 0$ and you have

$$F_x(x, y) \frac{dx}{dx} + F_y(x, y) \frac{dy}{dx} = 0.$$  

Now, if $F_y(x, y) \neq 0$, you can use the fact that $dx/dx = 1$ to conclude that

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)}.$$  

A similar procedure can be used to find the partial derivatives of functions of several variables that are defined implicitly.

**THEOREM 11.8 Chain Rule: Implicit Differentiation**

If the equation $F(x, y) = 0$ defines $y$ implicitly as a differentiable function of $x$, then

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)}, \quad F_y(x, y) \neq 0.$$  

If the equation $F(x, y, z) = 0$ defines $z$ implicitly as a differentiable function of $x$ and $y$, then

$$\frac{dz}{dx} = -\frac{F_x(x, y, z)}{F_y(x, y, z)} \quad \text{and} \quad \frac{dz}{dy} = -\frac{F_x(x, y, z)}{F_y(x, y, z)}, \quad F_x(x, y, z) \neq 0.$$  

**EXAMPLE 10 Finding a Derivative Implicitly**

Find $dy/dx$, given $y^3 + y^2 - 5y - x^2 + 4 = 0$.

**Solution** Begin by defining a function $F$ as $F(x, y) = y^3 + y^2 - 5y - x^2 + 4$. Then, using Theorem 11.8, you have

$$F_x(x, y) = -2x \quad \text{and} \quad F_y(x, y) = 3y^2 + 2y - 5$$

and it follows that

$$\frac{dy}{dx} = \frac{F_x(x, y)}{F_y(x, y)} = -\frac{2x}{3y^2 + 2y - 5}, \quad \frac{2x}{3y^2 + 2y - 5}.$$
**EXAMPLE 11** Finding Partial Derivatives Implicitly

Find \( \partial z/\partial x \) and \( \partial z/\partial y \), given \( 3x^2z - x^2y^2 + 2z^3 + 3yz - 5 = 0 \).

**Solution** To apply Theorem 11.8, let

\[
F(x, y, z) = 3x^2z - x^2y^2 + 2z^3 + 3yz - 5.
\]

Then

\[
F_x(x, y, z) = 6xz - 2xy^2 \\
F_y(x, y, z) = -2x^2y + 3z \\
F_z(x, y, z) = 3x^2 + 6z^2 + 3y
\]

and you obtain

\[
\frac{\partial z}{\partial x} = \frac{F_z(x, y, z)}{F_y(x, y, z)} = \frac{2xy^2 - 6xz}{3x^2 + 6z^2 + 3y}
\]

\[
\frac{\partial z}{\partial y} = \frac{F_z(x, y, z)}{F_x(x, y, z)} = \frac{2x^2y - 3z}{3x^2 + 6z^2 + 3y}
\]
Section 11.7

Extrema of Functions of Two Variables

- Find absolute and relative extrema of a function of two variables.
- Use the Second Partial Test to find relative extrema of a function of two variables.

Absolute Extrema and Relative Extrema

In Chapter 3, you studied techniques for finding the extreme values of a function of a single variable. In this section, you will extend these techniques to functions of two variables. For example, in Theorem 11.15, the Extreme Value Theorem for a function of a single variable is extended to a function of two variables.

Consider the continuous function \( f \) of two variables, defined on a closed bounded region \( R \). The values \( f(a, b) \) and \( f(c, d) \) such that

\[
\begin{align*}
  f(a, b) &\leq f(x, y) \leq f(c, d) \\
  (a, b) \text{ and } (c, d) &\text{ are in } R,
\end{align*}
\]

for all \((x, y)\) in \( R \) are called the minimum and maximum of \( f \) in the region \( R \), as shown in Figure 11.61. Recall from Section 11.2 that a region in the plane is closed if it contains all of its boundary points. The Extreme Value Theorem deals with a region in the plane that is both closed and bounded. A region in the plane is called bounded if it is a subregion of a closed disk in the plane.

**THEOREM 11.15 Extreme Value Theorem**

Let \( f \) be a continuous function of two variables \( x \) and \( y \) defined on a closed bounded region \( R \) in the \( xy \)-plane.

1. There is at least one point in \( R \) where \( f \) takes on a minimum value.
2. There is at least one point in \( R \) where \( f \) takes on a maximum value.

A minimum is also called an absolute minimum and a maximum is also called an absolute maximum. As in single-variable calculus, there is a distinction made between absolute extrema and relative extrema.

Definition of Relative Extrema

Let \( f \) be a function defined on a region \( R \) containing \((x_0, y_0)\).

1. The function \( f \) has a relative minimum at \((x_0, y_0)\) if \( f(x, y) \geq f(x_0, y_0) \) for all \((x, y)\) in an open disk containing \((x_0, y_0)\).
2. The function \( f \) has a relative maximum at \((x_0, y_0)\) if \( f(x, y) \leq f(x_0, y_0) \) for all \((x, y)\) in an open disk containing \((x_0, y_0)\).

To say that \( f \) has a relative maximum at \((x_0, y_0)\) means that the point \((x_0, y_0, z_0)\) is at least as high as all nearby points on the graph of \( z = f(x, y) \). Similarly, \( f \) has a relative minimum at \((x_0, y_0)\) if \((x_0, y_0, z_0)\) is at least as low as all nearby points on the graph. (See Figure 11.62.)
To locate relative extrema of $f$, you can investigate the points at which the gradient of $f$ is $0$ or the points at which one of the partial derivatives does not exist. Such points are called **critical points** of $f$.

**Definition of Critical Point**

Let $f$ be defined on an open region $R$ containing $(x_0, y_0)$. The point $(x_0, y_0)$ is a **critical point** of $f$ if one of the following is true.

1. $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$
2. $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ does not exist.

Recall from Theorem 11.11 that if $f$ is differentiable and

$$
\nabla f(x_0, y_0) = f_x(x_0, y_0)i + f_y(x_0, y_0)j
$$

then every directional derivative at $(x_0, y_0)$ must be $0$. This implies that the function has a horizontal tangent plane at the point $(x_0, y_0)$, as shown in Figure 11.63. It appears that such a point is a likely location of a relative extremum. This is confirmed by Theorem 11.16.

**THEOREM 11.16 Relative Extrema Occur Only at Critical Points**

If $f$ has a relative extremum at $(x_0, y_0)$ on an open region $R$, then $(x_0, y_0)$ is a critical point of $f$.

**EXPLORATION**

Use a graphing utility to graph

$$
z = x^3 - 3xy + y^3
$$

using the bounds $0 \leq x \leq 3$, $0 \leq y \leq 3$, and $-3 \leq z \leq 3$. This view makes it appear as though the surface has an absolute minimum. But does it?
**EXAMPLE 1** Finding a Relative Extremum

Determine the relative extrema of

\[ f(x, y) = 2x^2 + y^2 + 8x - 6y + 20. \]

**Solution** Begin by finding the critical points of \( f \). Because

\[ f_x(x, y) = 4x + 8 \quad \text{Partial with respect to } x \]

and

\[ f_y(x, y) = 2y - 6 \quad \text{Partial with respect to } y \]

are defined for all \( x \) and \( y \), the only critical points are those for which both first partial derivatives are 0. To locate these points, let \( f_x(x, y) \) and \( f_y(x, y) \) be 0, and solve the equations

\[ 4x + 8 = 0 \quad \text{and} \quad 2y - 6 = 0 \]

to obtain the critical point \((-2, 3)\). By completing the square, you can conclude that for all \( (x, y) \neq (-2, 3) \),

\[ f(x, y) = 2(x + 2)^2 + (y - 3)^2 + 3 > 3. \]

So, a relative minimum of \( f \) occurs at \((-2, 3)\). The value of the relative minimum is \( f(-2, 3) = 3 \), as shown in Figure 11.64.

Example 1 shows a relative minimum occurring at one type of critical point—the type for which both \( f_x(x, y) \) and \( f_y(x, y) \) are 0. The next example concerns a relative maximum that occurs at the other type of critical point—the type for which either \( f_x(x, y) \) or \( f_y(x, y) \) does not exist.

**EXAMPLE 2** Finding a Relative Extremum

Determine the relative extrema of \( f(x, y) = 1 - (x^2 + y^2)^{1/3} \).

**Solution** Because

\[ f_x(x, y) = -\frac{2x}{3(x^2 + y^2)^{2/3}} \quad \text{Partial with respect to } x \]

and

\[ f_y(x, y) = -\frac{2y}{3(x^2 + y^2)^{2/3}} \quad \text{Partial with respect to } y \]

it follows that both partial derivatives exist for all points in the \( xy \)-plane except for \((0, 0)\). Moreover, because the partial derivatives cannot both be 0 unless both \( x \) and \( y \) are 0, you can conclude that \((0, 0)\) is the only critical point. In Figure 11.65, note that \( f(0, 0) = 1 \). For all other \((x, y)\) it is clear that

\[ f(x, y) = 1 - (x^2 + y^2)^{1/3} < 1. \]

So, \( f \) has a relative maximum at \((0, 0)\).

**NOTE** In Example 2, \( f_x(x, y) = 0 \) for every point on the \( y \)-axis other than \((0, 0)\). However, because \( f_y(x, y) \) is nonzero, these are not critical points. Remember that one of the partials must not exist or both must be 0 in order to yield a critical point.
**The Second Partials Test**

Theorem 11.16 tells you that to find relative extrema you need only examine values of $f(x, y)$ at critical points. However, as is true for a function of one variable, the critical points of a function of two variables do not always yield relative maxima or minima. Some critical points yield saddle points, which are neither relative maxima nor relative minima.

As an example of a critical point that does not yield a relative extremum, consider the surface given by

$$f(x, y) = y^2 - x^2$$

as shown in Figure 11.66. At the point $(0, 0)$, both partial derivatives are 0. The function $f$ does not, however, have a relative extremum at this point because in any open disk centered at $(0, 0)$, the function takes on both negative values (along the $x$-axis) and positive values (along the $y$-axis). So, the point $(0, 0, 0)$ is a saddle point of the surface. (The term "saddle point" comes from the fact that the surface shown in Figure 11.66 resembles a saddle.)

For the functions in Examples 1 and 2, it was relatively easy to determine the relative extrema because each function was either given, or able to be written, in completed square form. For more complicated functions, algebraic arguments are less convenient and it is better to rely on the analytic means presented in the following Second Partials Test. This is the two-variable counterpart of the Second Derivative Test for functions of one variable. The proof of this theorem is left to a course in advanced calculus.

**THEOREM 11.17 Second Partials Test**

Let $f$ have continuous second partial derivatives on an open region containing a point $(a, b)$ for which

$$f_x(a, b) = 0 \quad \text{and} \quad f_y(a, b) = 0.$$

To test for relative extrema of $f$, consider the quantity

$$d = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2.$$

1. If $d > 0$ and $f_{xx}(a, b) > 0$, then $f$ has a relative minimum at $(a, b)$.
2. If $d > 0$ and $f_{xx}(a, b) < 0$, then $f$ has a relative maximum at $(a, b)$.
3. If $d < 0$, then $(a, b, f(a, b))$ is a saddle point.
4. The test is inconclusive if $d = 0$.

**NOTE** If $d > 0$, then $f_{xx}(a, b)$ and $f_{yy}(a, b)$ must have the same sign. This means that $f_{xx}(a, b)$ can be replaced by $f_{yy}(a, b)$ in the first two parts of the test.

A convenient device for remembering the formula for $d$ in the Second Partials Test is given by the $2 \times 2$ determinant

$$d = \begin{vmatrix} f_{xx}(a, b) & f_{xy}(a, b) \\ f_{yx}(a, b) & f_{yy}(a, b) \end{vmatrix}$$

where $f_{yx}(a, b) = f_{xy}(a, b)$ by Theorem 11.3.
**EXAMPLE 3** Using the Second Partialis Test

Find the relative extrema of $f(x, y) = -x^3 + 4xy - 2y^2 + 1$.

**Solution** Begin by finding the critical points of $f$. Because

$$f_x(x, y) = -3x^2 + 4y \quad \text{and} \quad f_y(x, y) = 4x - 4y$$

exist for all $x$ and $y$, the only critical points are those for which both first partial derivatives are 0. To locate these points, let $f_x(x, y)$ and $f_y(x, y)$ be 0 to obtain $-3x^2 + 4y = 0$ and $4x - 4y = 0$. From the second equation, you know that $x = y$ and, by substitution into the first equation, you obtain two solutions: $y = x = 0$ and $y = x = \frac{3}{2}$. Because

$$f_{xx}(x, y) = -6x, \quad f_{xy}(x, y) = -4, \quad \text{and} \quad f_{yy}(x, y) = 4$$

it follows that, for the critical point $(0, 0)$,

$$d = f_{xx}(0, 0)f_{yy}(0, 0) - [f_{xy}(0, 0)]^2 = 0 - 16 < 0$$

and, by the Second Partialis Test, you can conclude that $(0, 0, 1)$ is a saddle point of $f$. Furthermore, for the critical point $(\frac{3}{2}, \frac{3}{2})$,

$$d = f_{xx}(\frac{3}{2}, \frac{3}{2})f_{yy}(\frac{3}{2}, \frac{3}{2}) - [f_{xy}(\frac{3}{2}, \frac{3}{2})]^2$$

$$= -8(-4) - 16$$

$$= 16$$

$$> 0$$

and because $f_{xx}(\frac{3}{2}, \frac{3}{2}) = -8 < 0$, you can conclude that $f$ has a relative maximum at $(\frac{3}{2}, \frac{3}{2})$, as shown in Figure 11.67.

The Second Partialis Test can fail to find relative extrema in two ways. If either of the first partial derivatives does not exist, you cannot use the test. Also, if

$$d = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2 = 0$$

the test fails. In such cases, you can try a sketch or some other approach, as demonstrated in the next example.

**EXAMPLE 4** Failure of the Second Partialis Test

Find the relative extrema of $f(x, y) = x^2y^2$.

**Solution** Because $f_x(x, y) = 2xy^2$ and $f_y(x, y) = 2x^2y$, you know that both partial derivatives are 0 if $x = 0$ or $y = 0$. That is, every point along the $x$- or $y$-axis is a critical point. Moreover, because

$$f_{xx}(x, y) = 2y^2, \quad f_{xy}(x, y) = 2x^2, \quad \text{and} \quad f_{yy}(x, y) = 4xy$$

you know that if either $x = 0$ or $y = 0$, then

$$d = f_{xx}(x, y)f_{yy}(x, y) - [f_{xy}(x, y)]^2$$

$$= 4x^2y^2 - 16x^2y^2 = -12x^2y^2 = 0.$$
Absolute extrema of a function can occur in two ways. First, some relative extrema also happen to be absolute extrema. For instance, in Example 1, \( f(-2, 3) \) is an absolute minimum of the function. (On the other hand, the relative maximum found in Example 3 is not an absolute maximum of the function.) Second, absolute extrema can occur at a boundary point of the domain.

The concepts of relative extrema and critical points can be extended to functions of three or more variables. If all first partial derivatives of

\[
w = f(x_1, x_2, x_3, \ldots, x_n)
\]

exist, it can be shown that a relative maximum or minimum can occur at \((x_1, x_2, x_3, \ldots, x_n)\) only if every first partial derivative is 0 at that point. This means that the critical points are obtained by solving the following system of equations.

\[
\begin{align*}
    f_{x_1}(x_1, x_2, x_3, \ldots, x_n) &= 0 \\
    f_{x_2}(x_1, x_2, x_3, \ldots, x_n) &= 0 \\
    &\vdots \\
    f_{x_n}(x_1, x_2, x_3, \ldots, x_n) &= 0
\end{align*}
\]

The extension of Theorem 11.17 to three or more variables is also possible, although you will not consider such an extension in this text.

### Exercises for Section 11.7

In Exercises 1–6, identify any extrema of the function by recognizing its graph or form after completing the square. Verify your results by using the partial derivatives to locate any critical points and test for relative extrema. Use a computer algebra system to graph the function and label any extrema.

1. \( g(x, y) = (x - 1)^2 + (y - 3)^2 \)
2. \( g(x, y) = 9 - (x - 3)^2 - (y + 2)^2 \)
3. \( f(x, y) = \sqrt{x^2 + y^2 + 1} \)
4. \( f(x, y) = \sqrt{25 - (x - 2)^2 - y^2} \)
5. \( f(x, y) = x^2 + y^2 + 2x - 6y + 6 \)
6. \( f(x, y) = -x^2 - y^2 + 4x + 8y - 11 \)

In Exercises 7–16, examine the function for relative extrema.

7. \( f(x, y) = 2x^2 + 2xy + x^2 + 2x - 3 \)
8. \( f(x, y) = -x^2 - 5y^2 + 10x - 30y - 62 \)
9. \( f(x, y) = -5x^2 + 4xy - y^2 + 16x + 10 \)
10. \( f(x, y) = x^2 + 6xy + 10y^2 - 4y + 4 \)
11. \( z = 2x^2 + 3y^2 - 4x - 12y + 13 \)
12. \( z = -3x^2 - 2y^2 + 3x - 4y + 5 \)
13. \( f(x, y) = 2\sqrt{x^2 + y^2} + 3 \)
14. \( h(x, y) = (x^2 + y^2)^{1/3} + 2 \)
15. \( g(x, y) = 4 - |x| - |y| \)
16. \( f(x, y) = |x + y| - 2 \)

In Exercises 17–20, use a computer algebra system to graph the surface and locate any relative extrema and saddle points.

17. \( z = -4x \) \( x^2 + y^2 + 1 \)
18. \( f(x, y) = y^3 - 3xy^2 - 3y^2 - 3x^3 + 1 \)
19. \( z = (x^2 + 4y^2)e^{(-x^2 - y^2)} \)
20. \( z = e^y \)

In Exercises 21–28, examine the function for relative extrema and saddle points.

21. \( h(x, y) = x^2 - y^2 - 2x - 4y - 4 \)
22. \( g(x, y) = 120x + 120y - xy - x^2 - y^2 \)
23. \( h(x, y) = x^2 - 3xy - y^2 \)
24. \( g(x, y) = xy \)
25. \( f(x, y) = x^3 - 3xy + y^3 \)
26. \( z = e^{-x} \sin y \)
27. \( f(x, y) = 2xy - \frac{1}{2}(x^2 + y^2) + 1 \)
28. \( z = \left(\frac{1}{2} - x^2 - y^2\right)e^{(x^2 - y^2)} \)

In Exercises 29 and 30, examine the function for extrema without using the derivative tests, and use a computer algebra system to graph the surface. (Hint: By observation, determine if it is possible for \( z \) to be negative. When is \( z \) equal to 0?)

29. \( z = \frac{(x - y)^2}{x^2 - y^2} \)
30. \( z = \frac{(x - y)^2}{y^2 + y^2} \)

Think About It In Exercises 31–34, determine whether there is a relative maximum, a relative minimum, a saddle point, or insufficient information to determine the nature of the function \( f(x, y) \) at the critical point \((x_0, y_0)\).

31. \( f_{x_0}(x_0, y_0) = 9, \ f_{y_0}(x_0, y_0) = 4, \ f_{x_0}(x_0, y_0) = 6 \)
32. \( f_{x_0}(x_0, y_0) = -3, \ f_{y_0}(x_0, y_0) = -8, \ f_{x_0}(x_0, y_0) = 2 \)
33. \( f_{x_0}(x_0, y_0) = -9, \ f_{y_0}(x_0, y_0) = 9, \ f_{x_0}(x_0, y_0) = 4 \)
34. \( f_{x_0}(x_0, y_0) = 25, \ f_{y_0}(x_0, y_0) = 8, \ f_{x_0}(x_0, y_0) = 10 \)
**StudY Tip**
In Example 4, you can check that the two products are substitutes by observing that \( x_1 \) increases as \( p_2 \) increases and \( x_2 \) increases as \( p_1 \) increases.

**Application of Extrema**

**Example 4** Finding a Maximum Profit

A company makes two substitute products whose demand functions are given by

\[
\begin{align*}
  x_1 &= 200(p_2 - p_1) \\
  x_2 &= 500 + 100p_1 - 180p_2
\end{align*}
\]

where \( p_1 \) and \( p_2 \) are the prices per unit (in dollars) and \( x_1 \) and \( x_2 \) are the numbers of units sold. The costs of producing the two products are $0.50 and $0.75 per unit, respectively. Find the prices that will yield a maximum profit.

**Solution** The cost and revenue functions are as shown.

\[
\begin{align*}
  C &= 0.5x_1 + 0.75x_2 \\
  &= 0.5(200)(p_2 - p_1) + 0.75(500 + 100p_1 - 180p_2) \\
  &= 375 - 25p_1 - 35p_2 \\
  R &= p_1x_1 + p_2x_2 \\
  &= p_1(200(p_2 - p_1)) + p_2(500 + 100p_1 - 180p_2) \\
  &= -200p_1^2 + 180p_2^2 + 300p_1p_2 + 500p_2
\end{align*}
\]

This implies that the profit function is

\[
P = R - C = -200p_1^2 + 180p_2^2 + 300p_1p_2 + 500p_2 - (375 - 25p_1 - 35p_2)
\]

The maximum profit occurs when the two first partial derivatives are zero.

\[
\begin{align*}
  \frac{\partial P}{\partial p_1} &= -400p_1 + 300p_2 + 25 = 0 \\
  \frac{\partial P}{\partial p_2} &= -200p_1^2 + 180p_2^2 + 300p_1 + 35p_2 + 535 = 0
\end{align*}
\]

By solving this system simultaneously, you can conclude that the solution is \( p_1 = 3.14 \) and \( p_2 = 4.10 \). From the graph of the function shown in Figure 7.33, you can see that this critical number yields a maximum. So, the maximum profit is

\[
P(3.14, 4.10) = 761.48.
\]

**CheckPoint 4**

Find the prices that will yield a maximum profit for the products in Example 4 if the costs of producing the two products are $0.75 and $0.50 per unit, respectively.
**Example 5** Finding a Maximum Volume

Consider all possible rectangular boxes that are resting on the xy-plane with one vertex at the origin and the opposite vertex in the plane $6x + 4y + 3z = 24$, as shown in Figure 7.34. Of all such boxes, which has the greatest volume?

**SOLUTION** Because one vertex of the box lies in the plane given by $6x + 4y + 3z = 24$ or $z = \frac{1}{3}(24 - 6x - 4y)$, you can write the volume of the box as

$$V = xyz = x\left(\frac{1}{3}(24 - 6x - 4y)\right) = \frac{1}{3}(24xy - 6x^2y - 4xy^2).$$

To find the critical numbers, set the first partial derivatives equal to zero.

$$V_x = \frac{1}{3}(24y - 12xy - 4y^2) = \frac{1}{3}y(24 - 12x - 4y) = 0 \quad \text{Partial with respect to } x$$

$$V_y = \frac{1}{3}(24x - 6x^2 - 8xy) = \frac{1}{3}x(24 - 6x - 8y) = 0 \quad \text{Partial with respect to } y$$

The four solutions of this system are $(0, 0), (0, 6), (4, 0)$, and $(\frac{3}{2}, 2)$. Using the Second-Partials Test, you can determine that the maximum volume occurs when the width is $x = \frac{3}{2}$ and the length is $y = 2$. For these values, the height of the box is

$$z = \frac{1}{3}[24 - 6\left(\frac{3}{2}\right) - 4(2)] = \frac{8}{3}.$$

So, the maximum volume is

$$V = xyz = \left(\frac{3}{2}\right)(2)(\frac{8}{3}) = \frac{64}{9} \text{ cubic units.}$$

**CHECKPOINT 5**

Find the maximum volume of a box that is resting on the xy-plane with one vertex at the origin and the opposite vertex in the plane $2x + 4y + z = 8$.  

---

**CONCEPT CHECK**

1. Given a function of two variables $f$, state how you can determine whether $(x_0, y_0)$ is a critical point of $f$.
2. The point $(a, b, f(a, b))$ is a saddle point if what is true?
3. If $d > 0$ and $f_{xx}(a, b) > 0$, then what does $f$ have at $(a, b)$: a relative minimum or a relative maximum?
4. If $d > 0$ and $f_{xx}(a, b) < 0$, then what does $f$ have at $(a, b)$: a relative minimum or a relative maximum?