Section 7.6

Lagrange Multipliers

- Use Lagrange multipliers with one constraint to find extrema of functions of several variables and to answer questions about real-life situations.
- Use Lagrange multipliers with two constraints to find extrema of functions of several variables.

Lagrange Multipliers with One Constraint

In Example 5 in Section 7.5, you were asked to find the dimensions of the rectangular box of maximum volume that would fit in the first octant beneath the plane

$$6x + 4y + 3z = 24$$

as shown again in Figure 7.35. Another way of stating this problem is to say that you are asked to find the maximum of

$$V = xyz$$

subject to the constraint

$$6x + 4y + 3z - 24 = 0.$$  

This type of problem is called a constrained optimization problem. In Section 7.5, you answered this question by solving for $z$ in the constraint equation and then rewriting $V$ as a function of two variables.

In this section, you will study a different (and often better) way to solve constrained optimization problems. This method involves the use of variables called Lagrange multipliers, named after the French mathematician Joseph Louis Lagrange (1736–1813).

STUDY TIP

When using the Method of Lagrange Multipliers for functions of three variables, $F$ has the form

$$F(x, y, z, \lambda) = f(x, y, z) - \lambda g(x, y, z).$$

The system of equations used in Step 1 are as follows.

$$F_x(x, y, z, \lambda) = 0$$  
$$F_y(x, y, z, \lambda) = 0$$  
$$F_z(x, y, z, \lambda) = 0$$  
$$F_{\lambda}(x, y, z, \lambda) = 0$$

The Method of Lagrange Multipliers gives you a way of finding critical points but does not tell you whether these points yield minima, maxima, or neither. To make this distinction, you must rely on the context of the problem.
Example 1 Using Lagrange Multipliers: One Constraint

Find the maximum of

\[ V = xyz \]

subject to the constraint

\[ 6x + 4y + 3z - 24 = 0. \]

**SOLUTION** First, let \( f(x, y, z) = xyz \) and \( g(x, y, z) = 6x + 4y + 3z - 24 \). Then, define a new function \( F \) as

\[
F(x, y, z, \lambda) = f(x, y, z) - \lambda g(x, y, z) = xyz - \lambda(6x + 4y + 3z - 24).
\]

To find the critical numbers of \( F \), set the partial derivatives of \( F \) with respect to \( x, y, z, \) and \( \lambda \) equal to zero and obtain

\[
\begin{align*}
F_x(x, y, z, \lambda) &= yz - 6\lambda = 0 \\
F_y(x, y, z, \lambda) &= xz - 4\lambda = 0 \\
F_z(x, y, z, \lambda) &= xy - 3\lambda = 0 \\
F_\lambda(x, y, z, \lambda) &= -6x - 4y - 3z + 24 = 0.
\end{align*}
\]

Solving for \( \lambda \) in the first equation and substituting into the second and third equations produces the following.

\[
\begin{align*}
xz - 4\left(\frac{yz}{6}\right) &= 0 & \Rightarrow & & y &= \frac{3}{2}x \\
xy - 3\left(\frac{yz}{6}\right) &= 0 & \Rightarrow & & z &= 2x
\end{align*}
\]

Next, substitute for \( y \) and \( z \) in the equation \( F_x(x, y, z, \lambda) = 0 \) and solve for \( x \).

\[
F_x(x, y, z, \lambda) = 0
\]

\[
-6x - 4\left(\frac{3}{2}x\right) - 3(2x) + 24 = 0
\]

\[
-18x = -24
\]

\[
x = \frac{4}{3}
\]

Using this \( x \)-value, you can conclude that the critical values are \( x = \frac{4}{3}, y = 2, \) and \( z = \frac{8}{3} \), which implies that the maximum is

\[
V = \frac{4}{3} \cdot 2 \cdot \frac{8}{3} = \frac{64}{9} \text{ cubic units.}
\]

**CHECKPOINT 1**

Find the maximum volume of \( V = xyz \) subject to the constraint

\[ 2x + 4y + z - 8 = 0. \]
Example 2

Making a Decision

Finding a Maximum Production Level

A manufacturer's production is modeled by the Cobb-Douglas function

\[ f(x, y) = 100x^{3/4}y^{1/4} \]

Objective function

where \( x \) represents the units of labor and \( y \) represents the units of capital. Each labor unit costs $150 and each capital unit costs $250. The total expenses for labor and capital cannot exceed $50,000. Will the maximum production level exceed 16,000 units?

**SOLUTION**

Because total labor and capital expenses cannot exceed $50,000, the constraint is

\[ 150x + 250y = 50,000 \]

To find the maximum production level, begin by writing the function

\[ F(x, y, \lambda) = 100x^{3/4}y^{1/4} - \lambda(150x + 250y - 50,000) \]

Next, set the partial derivatives of this function equal to zero.

\[
\begin{align*}
F_x(x, y, \lambda) &= 75x^{-1/4}y^{1/4} - 150\lambda = 0 \\
F_y(x, y, \lambda) &= 25x^{3/4}y^{-3/4} - 250\lambda = 0 \\
F_\lambda(x, y, \lambda) &= -150x - 250y + 50,000 = 0
\end{align*}
\]

The strategy for solving such a system must be customized to the particular system. In this case, you can solve for \( \lambda \) in the first equation, substitute into the second equation, solve for \( x \), substitute into the third equation, and solve for \( y \).

\[
\begin{align*}
75x^{-1/4}y^{1/4} - 150\lambda &= 0 \\
\lambda &= \frac{1}{2}x^{-1/4}y^{1/4} \\
25x^{3/4}y^{-3/4} - 250(\frac{1}{2})x^{-1/4}y^{1/4} &= 0 \\
25x - 125y &= 0 \\
x &= 5y \\
-150(5y) - 250y + 50,000 &= 0 \\
-1000y &= -50,000 \\
y &= 50
\end{align*}
\]

Using this value for \( y \), it follows that \( x = 5(50) = 250 \). So, the maximum production level of

\[ f(250, 50) = 100(250)^{3/4}(50)^{1/4} \]

is 16,719 units

occurs when \( x = 250 \) units of labor and \( y = 50 \) units of capital. Yes, the maximum production level will exceed 16,000 units.

**Checkpoint 2**

In Example 2, suppose that each labor unit costs $200 and each capital unit costs $250. Find the maximum production level if labor and capital cannot exceed $50,000.
Economists call the Lagrange multiplier obtained in a production function the marginal productivity of money. For instance, in Example 2, the marginal productivity of money when \( x = 250 \) and \( y = 50 \) is

\[
\lambda = \frac{1}{2}x^{-1/4}y^{1/4} = \frac{1}{2}(250)^{-1/4}(50)^{1/4} = 0.334.
\]

This means that if one additional dollar is spent on production, approximately 0.334 additional unit of the product can be produced.

**Example 3** Finding a Maximum Production Level

In Example 2, suppose that \$70,000 is available for labor and capital. What is the maximum number of units that can be produced?

**SOLUTION** You could rework the entire problem, as demonstrated in Example 2. However, because the only change in the problem is the availability of additional money to spend on labor and capital, you can use the fact that the marginal productivity of money is

\[
\lambda \approx 0.334.
\]

Because an additional \$20,000 is available and the maximum production in Example 2 was 16,719 units, you can conclude that the maximum production is now

\[
16,719 + (0.334)(20,000) \approx 23,400 \text{ units}.
\]

Try using the procedure demonstrated in Example 2 to confirm this result.

**CHECKPOINT 3**

In Example 3, suppose that \$80,000 is available for labor and capital. What is the maximum number of units that can be produced?
In Example 4 in Section 7.5, you found the maximum profit for two substitute products whose demand functions are given by

\[ x_1 = 200(p_2 - p_1) \quad \text{Demand for product 1} \]
\[ x_2 = 500 + 100p_1 - 180p_2 \quad \text{Demand for product 2} \]

With this model, the total demand, \( x_1 + x_2 \), is completely determined by the prices \( p_1 \) and \( p_2 \). In many real-life situations, this assumption is too simplistic; regardless of the prices of the substitute brands, the annual total demands for some products, such as toothpaste, are relatively constant. In such situations, the total demand is limited, and variations in price do not affect the total demand as much as they affect the market share of the substitute brands.

**Example 4** Finding a Maximum Profit

A company makes two substitute products whose demand functions are given by

\[ x_1 = 200(p_2 - p_1) \quad \text{Demand for product 1} \]
\[ x_2 = 500 + 100p_1 - 180p_2 \quad \text{Demand for product 2} \]

where \( p_1 \) and \( p_2 \) are the prices per unit (in dollars) and \( x_1 \) and \( x_2 \) are the numbers of units sold. The costs of producing the two products are \$0.50 and \$0.75 per unit, respectively. The total demand is limited to 200 units per year. Find the prices that will yield a maximum profit.

**SOLUTION** From Example 4 in Section 7.5, the profit function is modeled by

\[ P = -200p_1^2 - 180p_2^2 + 200p_1p_2 + 25p_1 + 535p_2 - 375. \]

The total demand for the two products is

\[ x_1 + x_2 = 200(p_2 - p_1) + 500 + 100p_1 - 180p_2 \]
\[ = -100p_1 + 20p_2 + 500. \]

Because the total demand is limited to 200 units,

\[ -100p_1 + 20p_2 + 500 = 200. \]

Using Lagrange multipliers, you can determine that the maximum profit occurs when \( p_1 = \$3.94 \) and \( p_2 = \$4.69 \). This corresponds to an annual profit of \$712.21.

**Checkpoint 4**

In Example 4, suppose the total demand is limited to 250 units per year. Find the prices that will yield a maximum profit.

![Figure 7.36](image)

**Study Tip**

The constrained optimization problem in Example 4 is represented graphically in Figure 7.36. The graph of the objective function is a paraboloid and the graph of the constraint is a vertical plane. In the "unconstrained" optimization problem on page 520, the maximum profit occurred at the vertex of the paraboloid. In this "constrained" problem, however, the maximum profit corresponds to the highest point on the curve that is the intersection of the paraboloid and the vertical "constraint" plane.
Lagrange Multipliers with Two Constraints

In Examples 1 through 4, each of the optimization problems contained only one constraint. When an optimization problem has two constraints, you need to introduce a second Lagrange multiplier. The customary symbol for this second multiplier is \( \lambda \), the Greek letter lambda.

**Example 5** Using Lagrange Multipliers: Two Constraints

Find the minimum value of
\[
  f(x, y, z) = x^2 + y^2 + z^2
\]
subject to the constraints
\[
  x + y - 3 = 0 \quad \text{Constraint 1}
\]
\[
  x + z - 5 = 0 \quad \text{Constraint 2}
\]

**SOLUTION** Begin by forming the function
\[
  F(x, y, z, \lambda, \mu) = x^2 + y^2 + z^2 - \lambda(x + y - 3) - \mu(x + z - 5).
\]
Next, set the five partial derivatives equal to zero, and solve the resulting system of equations for \( x, y, \) and \( z \).

\[
  F_x(x, y, z, \lambda, \mu) = 2x - \lambda = 0 \quad \text{Equation 1}
\]
\[
  F_y(x, y, z, \lambda, \mu) = 2y = 0 \quad \text{Equation 2}
\]
\[
  F_z(x, y, z, \lambda, \mu) = 2z - \mu = 0 \quad \text{Equation 3}
\]
\[
  F_\lambda(x, y, z, \lambda, \mu) = -x - y + 3 = 0 \quad \text{Equation 4}
\]
\[
  F_\mu(x, y, z, \lambda, \mu) = -x - z + 5 = 0 \quad \text{Equation 5}
\]

Solving this system of equations produces \( x = \frac{8}{3}, y = \frac{1}{3}, \) and \( z = \frac{7}{3} \). So, the minimum value of \( f(x, y, z) \) is
\[
  f\left(\frac{8}{3}, \frac{1}{3}, \frac{7}{3}\right) = \left(\frac{8}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{7}{3}\right)^2 = \frac{38}{3}.
\]

**CHECKPOINT 5**

Find the minimum value of
\[
  f(x, y, z) = x^2 + y^2 + z^2
\]
subject to the constraints
\[
  x + y - 2 = 0
\]
\[
  x + z - 4 = 0.
\]

**CONCEPT CHECK**

1. Lagrange multipliers are named after what French mathematician?
2. What do economists call the Lagrange multiplier obtained in a production function?
3. Explain what is meant by constrained optimization problems.
4. Explain the Method of Lagrange Multipliers for solving constrained optimization problems.